A CHARACTERIZATION OF THE CLOSED 2-CELL*

BY

HASSLER WHITNEY†

1. **Introduction.** A number of characterizations have been given of the simple closed surface.‡ The proofs involve considerable point set difficulties. We give here a characterization of the closed 2-cell, that is, a point set homeomorphic with a circle and its interior. The fundamental theorem is partly of a combinatorial and partly of a continuity nature. It reads

THEOREM I. *Let R be a continuous curve* § *containing the simple closed curve J, such that*

(1) *J is irreducibly homologous to zero in R, and*

(2) *If γ is an arc with just its two end points a and b on J, then $R-\gamma$ is not connected.*

Let R' and J' be defined similarly. Then R and R' are homeomorphic, with J corresponding with J'.

That R is a closed 2-cell then follows immediately from the following theorem. We note that J corresponds with the circle, that is, J is the boundary of R.

THEOREM II. *If I is a circle in the plane and S is I with its interior, then S and I satisfy the conditions prescribed for R and J in the above theorem.*

The exact meaning of Condition (1) of Theorem I is given in §4; a stronger condition is the following: For every $\epsilon>0$ and any two points a and b on J, there is a set of points a_{ij} in R, $1\leqq i\leqq m$, $1\leqq j\leqq n$, such that all points a_{1j} coincide with a, all points a_{mj} coincide with b, all points a_{i1} lie on one arc ab of J, all points a_{in} lie on the other arc ab of J, and||

$$\rho(a_{ij}, a_{i+1,j}) < \epsilon, \quad \rho(a_{ij}, a_{i,j+1}) < \epsilon;$$

moreover, this does not hold in any proper subset of R containing J.

* Presented to the Society, October 31, 1931; received by the editors April 13, 1932.

† National Research Fellow.

‡ That is, a point set homeomorphic with the surface of a sphere. See L. Zippin, American Journal of Mathematics, vol. 52 (1931), pp. 331–350; these Transactions, vol. 31 (1929), pp. 744–770; C. Kuratowski, Fundamenta Mathematicae, vol. 13 (1929), pp. 307–318; also references in these papers.

§ See Lemma A.

|| $\rho(p, q)$ = distance from p to q, or in general, distance between two point sets; $\delta(S)$ = diameter of S; $V_\epsilon(S)$ = those points p for which $\rho(p, S)<\epsilon$; $W_\epsilon(S)$ = those points p for which $\rho(p, S)\leqq\epsilon$.

Notations and preliminary theorems are given in §§2, 3 and 4; an outline of the proof of Theorem I will be found in §5. The Jordan and related theorems follow of course from the above theorems.

2. **Point set background.** Elementary properties of point sets we shall need may be found in Hausdorff, *Mengenlehre*, chapter VI. A continuous curve is a metric space which can be expressed as the continuous image of a closed line segment. An arc is the topological image of a closed line segment; a simple closed curve, the topological image of a circle.

Two fundamental lemmas are the following:

LEMMA A.* *A compact, connected and locally connected metric space is a continuous curve, and conversely.*

LEMMA B.† *A continuous curve is arcwise connected.*

That is, any two points p and q in the set are end points of an arc pq in the set. Using the definition of a continuous curve, it is easily seen that two continuous curves which have common points form a continuous curve.

From these lemmas we deduce the following known theorems.

LEMMA C. *Any continuum C of diameter $<\epsilon$ in a continuous curve R is contained in a continuous curve C' in R of diameter $<\epsilon$.*

Say $\delta(C)=\epsilon-\epsilon'$. R being the continuous image of a closed line segment, we can divide this segment into segments so small that the diameter of the image of each is $<\epsilon'$. We let C' be the union of all of these images which have points in common with C.

LEMMA D. *A continuous curve R is locally arcwise connected.*

That is, given a point p and an $\epsilon>0$, there is a $\delta>0$ such that if $q\subset V_\delta(p)$, then there is an arc pq in R of diameter $<\epsilon$. As R is locally connected, we can take δ so that if $q\subset V_\delta(p)$, there is a continuum C in R of diameter $<\epsilon$ containing p and q. The continuum C is contained in a continuous curve C' of diameter $<\epsilon$, and C' is arcwise connected; hence there is an arc $pq\subset C'\subset R$, and $\delta(pq)<\epsilon$.

R is of course uniformly locally arcwise connected, by the Borel Theorem.

LEMMA E. *A connected open subset R' of a continuous curve R is arcwise connected.*

---

* See G. T. Whyburn, *Concerning continuous images of the interval*, American Journal of Mathematics, vol. 53 (1931), pp. 670–674.

† See references in R. L. Moore, *Report on continuous curves*, Bulletin of the American Mathematical Society, vol. 29 (1923), p. 293, footnote (†).

If there are two points p and q in R' which are joined by no arc in R', let A contain p and all points of R' joined to p by an arc in R', and put $B=R'-A$; then there is no arc joining a point of A to a point of B in R'. As R' is connected, there is a point p' in one of these sets, say B, which is a limit point of points of the other set, A. As R' is open in R, $\rho(p', R-R')=\epsilon>0$. We can take q' in A so close to p' that there is an arc $p'q'$ in R of diameter $<\epsilon$. But then $p'q' \subset R'$, a contradiction.

Suppose R is connected, and $p \subset R$ is such a point that $R-p$ is not connected. Then p is called a cut point of R.

LEMMA F. *Let R be a continuous curve without a cut point. Then for every $\epsilon>0$ there is a $\delta>0$ such that if $\rho(q, p) \geqq \epsilon$ and $\rho(q', p) \geqq \epsilon$, then there is an arc qq' with no points in $V_\delta(p)$.*

Suppose the contrary. Then there are three sequences of points $\{p_n\}$, $\{q_n\}$, $\{q_n'\}$, approaching points p, q, q', respectively, with $\rho(q_n, p_n) \geqq \epsilon$, $\rho(q_n', p_n) \geqq \epsilon$, and such that for each n, any arc $q_n q_n'$ must contain points in $V_{\delta_n}(p_n)$, where $\lim_{n\to\infty}\delta_n=0$. By Lemma D it is seen that for any n greater than some N there are arcs $q_n q$, $q_n' q'$, with no points in $V_{\epsilon/2}(p)$. It follows that any arc qq' must pass through p, contradicting Lemma E (as p is not a cut point).

3. **Combinatorial background.*** A *k-simplex*, or *abstract k-simplex*, is a set of k elements (say points) $a_1a_2 \cdots a_k$. The order in which we write the points is immaterial. For $k=0$, 1 and 2 we use also the terms *vertex*, *segment* and *triangle* respectively. A *k-chain* is a set of k-simplexes, and is written as the sum of these simplexes. The *sum* (mod 2) of several k-chains is the k-chain containing those simplexes which occur in an odd number of the k-chains.

The *boundary K* of a *k-simplex L*, $k>0$, is the sum of all $(k-1)$-simplexes formed by dropping out one of the vertices of the simplex. We write $L\to K$. A 0-simplex has no boundary. Thus

$$a \to 0,\ ab \to a + b,\ abc \to ab + ac + bc.$$

The *boundary* of a k-chain is the sum (mod 2) of the boundaries of the simplexes of the chain. Thus

$$ab + bc + cd \to a + d,\ abc + bcd \to ab + ac + bd + cd.$$

Evidently *the boundary of a sum of several k-chains is the sum of the boundaries of the chains.* If a k-chain has no boundary, it is called a k-cycle. (Any 0-chain is a 0-cycle.) *The boundary of a k-chain* ($k>0$) *is a* $(k-1)$*-cycle.* This is evi-

* Compare L. Vietoris, *Über den höheren Zusammenhang kompakter Räume*, Mathematische Annalen, vol. 97 (1927), pp. 454–472.

dent if the k-chain is a k-simplex. The general case then follows from the last theorem.

LEMMA G. *If $K\to a+b$ is a* 1-*chain, then there is a chain of segments aa_1, $a_1a_2, \cdots, a_nb$ in K.*

For otherwise we could divide the segments of K into two groups $K_1\supset a$ and $K_2\supset b$, no two simplexes from different groups having a common vertex. But then $K_1\to a$, $K_2\to b$, which cannot be, as the boundary of any 1-chain contains an even number of vertices.

A 1-circuit is a 1-cycle of the form $a_1a_2, a_2a_3, \cdots, a_{n-1}a_n, a_na_1$, the vertices being distinct except as shown.

LEMMA H. *Any* 1-*cycle K is a sum of* 1-*circuits.*

If a_1a_n is a segment of K, then $K+a_1a_n\to a_1+a_n$, as $K\to 0$ and $a_1a_n\to a_1+a_n$. We can thus find a set of distinct segments and vertices $a_1a_2, \cdots, a_{n-1}a_n$ in $K+a_1a_n$ not containing a_1a_n. This with a_1a_n is a 1-circuit K_1. As $K_1\to 0$, $K+K_1$ is a 1-cycle containing no segments of K_1, and it contains a 1-circuit K_2. Continuing, we find $K=K_1+K_2+\cdots+K_m$.

4. A k-chain K is said to *lie in a point set* R if each vertex of K is in R. Any vertex now has both a name and a position. Two vertices are distinct if their names are distinct, irrespective of whether they coincide in position or not. ϵ being a positive number, a k-simplex $K\subset R$ is called an (ϵ, k)-*simplex* in R if $\delta(K)<\epsilon$, i.e. if any two vertices of K are within ϵ of each other. A k-chain is an (ϵ, k)-*chain* if each of its simplexes is an (ϵ, k)-simplex. A k-cycle K in S is said to be ϵ-*homologous to zero* ($K\epsilon\sim 0$) in R if there is an $(\epsilon, k+1)$-chain L in R of which K is the boundary. *If $K_1\epsilon\sim 0$ and $K_2\ \epsilon\sim 0$, then K_1+K_2* $\epsilon\sim 0$. We write also $K_1\epsilon\sim K_2$ for $K_1+K_2\epsilon\sim 0$. *If $K_1\epsilon\sim K_2$ and $K_2\epsilon\sim K_3$, then $K_1\epsilon\sim K_3$.*

Suppose the closed set R contains the simple closed curve J. If for every $\epsilon>0$ there is a $\delta>0$ such that any $(\delta, 1)$-cycle on J is $\epsilon\sim 0$ in R, then we say that $J\sim 0$ *in* R. If J is ~ 0 in R but is not ~ 0 in any proper closed subset of R containing J, then we say that J is *irreducibly* ~ 0 in R.

LEMMA I. *Given a simple closed curve J, let us divide it into the arcs* $\overline{a_1a_2}$, $\overline{a_2a_3}, \cdots, \overline{a_{n-1}a_n}, \overline{a_na_1}$, each of diameter $<\epsilon/2$. Let δ be smaller than the distance between any two of these arcs which have no common points. Then if $K'=a_1a_2+a_2a_3+\cdots+a_na_1$ and K is any $(\delta, 1)$-cycle on J, K is either $\epsilon\sim 0$ or $\epsilon\sim K'$ on J.*

By Lemma H, K is a sum of 1-circuits $K_1, \cdots, K_m$. If we show that each

* Here, $\overline{a_1a_2}$ denotes an arc, and a_1a_2, a segment.

K_i is $\epsilon\sim\alpha_i K'$, $\alpha_i=0$ or 1, it will follow that $K=\sum K_i\ \epsilon\sim\sum\alpha_i K'=0$ or K' (depending on whether $\sum\alpha_i$ is even or odd), and the lemma will be proved.

Consider any $K_i=b_1b_2+b_2b_3+\cdots+b_sb_1$, say. If a vertex b_j of K_i does not lie on any point a_k, say $b_j\subset\overline{a_ka_{k+1}}$; add to K_i the boundary of the ϵ-triangles $b_{j-1}b_ja_k'+b_jb_{j+1}a_k'$, where a_k' is a new vertex lying on a_k. The result is an $(\epsilon, 1)$-circuit $K_i^{(1)}\ \epsilon\sim K_i$, the vertex b_j having been replaced by the vertex a_k'. Repeat the process till we have an $(\epsilon, 1)$-circuit $K''=c_1c_2+c_2c_3+\cdots+c_sc_1\ \epsilon\sim K_i$.

Now any two consecutive vertices c_j, c_{j+1} lie on the same or consecutive vertices of K'. Suppose c_j is on a_k and c_{j+2} is on a_{k+p}, $p\neq 2$ or -2. Then add the boundary of $c_jc_{j+1}c_{j+2}$, replacing the segments $c_jc_{j+1}+c_{j+1}c_{j+2}$ by the single segment c_jc_{j+2}. Continue till we arrive at a (possibly void) $(\epsilon, 1)$-circuit $K^*=d_1d_2+\cdots+d_rd_1\epsilon\sim K_i$. If d_1 lies on a_k, then d_{j+1} lies on $a_{k\pm j}$, where we put $n+p=p$, etc.

If K^* contains no segments, $K_i\ \epsilon\sim 0$. Otherwise, following the vertices $d_1, d_2, \cdots, d_r, d_1$ of K^*, we have gone around J p times say. Add to K^* the boundaries of all the $2r$ ϵ-triangles of the following sort. If d_j lies on a_k, and d_{j+1} on $a_{k\pm 1}$, two of the triangles are $d_jd_{j+1}a_k$ and $d_{j+1}\,a_ka_{k\pm 1}$. The result is an $(\epsilon, 1)$-cycle $pK'=0$ or K'. Thus $K_i\epsilon\sim 0$ or K', and the proof is complete.

An immediate consequence of this lemma is

Lemma J. *Let the simple closed curve J lie in the closed set R. If for every $\epsilon>0$ there is a 1-cycle K' in J as above described which is $\epsilon\sim 0$ in R, then $J\sim 0$ in R.*

Lemma K. *If γ is an arc, then for every $\epsilon>0$ there is a $\delta>0$ such that any $(\delta, 1)$-cycle on γ is $\epsilon\sim 0$ on γ.*

The proof below holds in fact if γ is a closed k-cell, any k. It is sufficient to prove it for the case that γ is a closed line segment, in which case we can take $\delta=\epsilon/2$.†

Let K be a $(\delta, 1)$-cycle on γ, let a_0b_0 be a segment of K, and say $\delta(\gamma)=\alpha$. Choose a fixed point p in γ, and an integer $n>\alpha/\delta$. Let the vertices $a_1, a_2, \cdots, a_{n-1}$ divide the segment a_0p into n equal parts, and similarly for the vertices $b_1, b_2, \cdots, b_{n-1}$. Add to K the boundaries of all triangles of the form $a_ia_{i+1}b_i$, $a_{i+1}b_ib_{i+1}$, $a_{n-1}b_{n-1}p$, and of all similar triangles corresponding to the other segments of K. The result is 0. As all the triangles employed are ϵ-triangles, $K\epsilon\sim 0$ in γ.

5. **Outline of the proof of Theorem I.** The proof runs as follows.

† The essential point in the proof below is that γ is convex: any two points of γ are end points of a line segment in γ. The proof is then easily extended to the case of any set homeomorphic with γ.

(a) In §6 we show how an arc γ can be drawn in R crossing J,† avoiding two given closed sets. $R-\gamma$ is not connected.

(b) In §7 we prove some lemmas. These show (§8) that $R-\gamma$ contains exactly two components A' and B'. If $A=A'+\gamma$, then A and its boundary curve J_A (which is γ plus a part of J) satisfy condition (1) of the theorem; similarly for $B=B'+\gamma$ and J_B. Further, A and B are continuous curves.

(c) In §9 it is shown that any arc in A (or B) crossing J_A (J_B) divides A (B). Thus A and J_A (B and J_B) satisfy all the conditions of the theorem. Hence we can cut up each set just as we cut up R, and can continue indefinitely.

(d) The object of §10 is to prove that R may be cut into pieces of arbitrarily small diameter.

(e) The homeomorphism between R and R' is now easily established. We cut R up indefinitely, and cut R' in a corresponding fashion. Any point p of R lies in a descending sequence of pieces; the corresponding sequence in R' determines a point p', which we let correspond to p.

We turn now to the detailed proof.

6. An arc crossing J. We prove here

LEMMA L.‡ *Let the simple closed curve J be ~ 0 in the continuous curve R. Let c and d be two points of J, dividing J into the two arcs η_1 and η_2. If C and D are two closed sets in R containing c and d respectively, and $C\cdot D=0$, then there is an arc γ in R joining η_1 to η_2 which has no points in C or in D.*

Say $\rho(C, D)=3\epsilon$, and put $C'=W_\epsilon(C)$, $D'=W_\epsilon(D)$; then $\rho(C', D')=\epsilon$. Take σ so small that any two points in R within σ of each other are joined by an arc of diameter $<\epsilon$ (Lemma D). Take δ so small that any $(\delta, 1)$-cycle on J is $\sigma\sim 0$ in R. Construct the $(\delta, 1)$-cycle $K=cc_1+c_1c_2+\cdots+c_md+dd_1+d_1d_2+\cdots+d_nc$, $c_i\subset\eta_1$, $d_i\subset\eta_2$. There is a $(\sigma, 2)$-chain

$$L = L_C + L_D \to K$$

in R, where we let L_C contain all those triangles of L with vertices in C', and let L_D be the rest of L.

Say

$$L_C \to K_C = K_C' + K^*,$$

where we let K_C' contain all those segments of K_C which are also in K. As $L_C\subset V_\sigma(C')$, $K^*\cdot D'=0$. Define K_D' by the relation

† That is, γ lies in R, and has only its end points on J.

‡ Compare P. Urysohn, *Über Räume mit verschwindender erster Brouwerscher Zahl*, Proceedings Amsterdam Akademie van Wetenschappen, vol. 31 (1928), pp. 808–810.

$$L_D \rightarrow K_D = K_D' + K^*.$$

Adding these relations gives L on the left, and hence K on the right:

$$K = K_C' + K_D'.$$

As all the segments of K_C' are in K, K_D' must contain just those segments of K not in K_C'; in particular, it contains no segments of K^*. Hence all the segments of K^* are present in $K_D' + K^*$, the boundary of L_D (i.e. none have canceled out with segments of K_D'). Hence, as $L_D \cdot C' = 0$,

$$K^* \cdot C' = K^* \cdot D' = 0.$$

As K_C is the boundary of L_C, it is a 1-cycle; hence

$$K_C + cc_1 = K_C' + K^* + cc_1 \rightarrow c + c_1.$$

By Lemma G, $K_C + cc_1$ contains a chain of segments joining c_1 to c. Following this chain, let p_s be the first vertex in η_2, and p_0, the last vertex before p_s in η_1, and say $p_0p_1, p_1p_2, \cdots, p_{s-1}p_s$ are the segments in between. We shall show that these segments are in K^*. If $s>1$ this is obvious, as then $p_1, \cdots p_{s-1}$ exist and are not on J. Suppose $s=1$ and p_0p_1 is not in K^*; then it is in $K_C' + cc_1$. It could only be the segment cc_1. But cc_1 lies in K and not in K_D, hence it is in K_C; it is not in K^*, hence it is in $K_C + K^* = K_C'$, and therefore not in $K_C' + cc_1$. This proves the statement.

Now let $\overline{p_ip_{i+1}}$ be an arc of diameter $<\epsilon$ in R, $i=0, \cdots, s-1$. These arcs form a continuous curve, from which we can pick out an arc γ (Lemma B) joining η_1 to η_2; we can take γ so only its end points are on J. As $p_ip_{i+1} \subset K^*$ and $\delta(\overline{p_ip_{i+1}}) < \epsilon$, γ has no points in C or in D, and the lemma is proved.

7. We prove three lemmas.

LEMMA M. *If $J \subset C$, $J \sim 0$ in $C+D$, and $C \cdot D =$ an arc γ, then $J \sim 0$ in C.*

Given an $\epsilon > 0$, choose first ϵ_1 so small that any $(3\epsilon_1, 1)$-cycle on γ is $\epsilon \sim 0$ in γ (Lemma K). Take next $\epsilon_2 < \epsilon_1$ so that if $p \subset D$ and $\rho(p, C) < \epsilon_2$, then $\rho(p, \gamma) < \epsilon_1$. (If $D_1 = D - D \cdot V_{\epsilon_1}(\gamma)$, take $\epsilon_2 < \rho(D_1, C)$.) Take finally $\delta < \epsilon_2$ so that any $(\delta, 1)$-cycle K on J is $\epsilon_2 \sim 0$ in $C+D$; we shall show that $K \epsilon \sim 0$ in C.

Let $L \rightarrow K$ be an $(\epsilon_2, 2)$-chain in $C+D$. Take any vertex p of L in $D \cdot V_{\epsilon_2}(C) - \gamma$, and replace it by a vertex $p' \subset \gamma$, where $\rho(p, p') < \epsilon_1$. L is thus replaced by a $(3\epsilon_1, 2)$-chain L', in which each triangle lies wholly in either C or D. Moreover, $L' \rightarrow K$, as no vertices of K have been moved.

Put $L' = L_C + L_D$, where L_C contains those triangles of L' in C. Say

$$L_C \rightarrow K + K^*; \text{ then } L_D \rightarrow K^*.$$

K^* is a $(3\epsilon_1, 1)$-cycle lying in $C \cdot D = \gamma$; it bounds an $(\epsilon, 2)$-chain L^* in γ. Hence

$$L_C + L^* \to (K + K^*) + K^* = K.$$

L_C+L^* is an $(\epsilon, 2)$-chain in C, and the lemma is proved.

LEMMA N. *Let $A \cdot B = \gamma$, an arc whose end points are a and b. Let the arcs α and β join a and b in A and B respectively, neither having any points other than a and b in common with γ. If $\alpha+\beta\sim 0$ in $A+B$, then $\alpha+\gamma\sim 0$ in A.*

Given an $\epsilon>0$, choose ϵ_1, ϵ_2 and δ as in the last lemma. Take $(\delta, 1)$-chains K_α, K_β and K_γ in α, β and γ respectively, each bounded by $a+b$; by Lemma J, it is sufficient to show that $K_\alpha+K_\gamma$ $\epsilon\sim 0$ in A.

$K_\alpha+K_\beta$ bounds an $(\epsilon_2, 2)$-chain L in $A+B$; we move each vertex of L in $B\cdot V_{\epsilon_2}(A)-\gamma$ onto γ, giving a $(3\epsilon_1, 2)$-chain $L'\to K_\alpha+K_\beta'$. Say $L'=L_A+L_B$, where $L_A\subset A$, $L_B\subset B$. If $L_A\to K_\alpha+K^*$, then $L_B\to K_\beta'+K^*$, and $K^*\subset\gamma$. K^*+K_γ is a $(3\epsilon_1, 1)$-cycle on γ bounding an $(\epsilon, 2)$-chain L^* in γ. Hence $L_A+L^*\to K_\alpha+K_\gamma$ in A, completing the proof.

LEMMA O. *Let α, β and γ be three arcs such that $\alpha\cdot\beta=\alpha\cdot\gamma=\beta\cdot\gamma=a+b$. Say $\alpha+\gamma\subset A$ and $\beta+\gamma\subset B$. If $\alpha+\gamma\sim 0$ in A and $\beta+\gamma\sim 0$ in B, then $\alpha+\beta\sim 0$ in $A+B$.*

Define K_α, K_β, K_γ as before; we need merely show that $K_\alpha+K_\beta\,\epsilon\sim 0$ in $A+B$. There are $(\epsilon, 2)$-chains L_A and L_B such that $L_A\to K_\alpha+K_\gamma$ in A and $L_B\to K_\beta+K_\gamma$ in B; hence $L_A+L_B\to K_\alpha+K_\beta$ in $A+B$.

8. **The set $R-\gamma$.** Let γ be any arc in R crossing J; say the end points of γ divide J into the two arcs α and β. By condition (2) of the theorem, $R-\gamma$ is not connected. Let A' and B' be those components of $R-\gamma$ containing $\langle\alpha\rangle$† and $\langle\beta\rangle$ respectively. These are not the same component. For if they were, putting $A = A'+\gamma, D = R - A'$, we have $J\subset A$, $J\sim 0$ in $R=A+D$, and $A\cdot D=\gamma$; hence, by Lemma M, $J\sim 0$ in A, a proper subset of R, contrary to condition (1) of the theorem.

The same reasoning shows that R has no cut point p; we need merely replace γ by p in Lemma M and above.

Put

$$A = A' + \gamma,\ B = B' + \gamma.$$

If $D=R-A'$, then $A\cdot D=\gamma$ and $J=\alpha+\beta\sim 0$ in $R=A+D$. Hence, by Lemma N, $\alpha+\gamma\sim 0$ in A. Similarly, $\beta+\gamma\sim 0$ in B. Consequently, by Lemma O, $J\sim 0$ in $A+B$, from which follows that $A+B=R$.

Moreover, $\alpha+\gamma$ is irreducibly ~ 0 in A. For if $\alpha+\gamma\sim 0$ in A^*, $\alpha+\gamma\subset A^*\subset A$, then, by Lemma O, $\alpha+\beta\sim 0$ in A^*+B; hence $A^*+B=R$, which is only possible if $A^*=A$. Similarly, $\beta+\gamma$ is irreducibly ~ 0 in B.

† $\langle\alpha\rangle$ is α except for its end points, etc.

Let us show that A is a continuous curve. It is connected, as A' is; it is self-compact, being a closed subset of a compact space. A is locally connected. For if p and q are points of A close enough together, there is an arc pq in R of small diameter; if pq lies partly in B', we can replace that part of it by an arc of γ of small diameter. Lemma A now applies. Similarly, B is a continuous curve.

9. We shall now show that any arc δ crossing $J_A=\alpha+\gamma$ in A divides A. The following two lemmas will be useful.

LEMMA P. *If η_1 and η_2 are arcs contained within the arcs γ and β respectively, then there is an arc pq crossing $J_B=\beta+\gamma$ in B, with $p\subset\eta_1$, $q\subset\eta_2$.*

This is an immediate consequence of Lemma L, if we take, for the closed sets of that lemma, the closed intervals of J_B complementary to η_1 and η_2.

LEMMA Q. *There are no two arcs ab and cd in R without common points, each crossing J, whose end points are in the order $acbd$ on J.*

This follows directly from what we have seen above.

To show that δ divides A, we must consider four cases.

Case 1. Both end points of δ lie on α. Suppose $A-\delta$ is connected; then it is arcwise connected, by Lemma E. Hence there is an arc in $A-\delta$ joining a point p of α lying between the two end points of δ and a point q within γ. If η_1 is an arc within γ containing q, there is an arc rs in B joining η_1 to a point s within β, with only its end points r and s on J_B, by Lemma P. The arc $pqrs$ crosses J and does not touch δ. But the end points of this arc alternate with those of δ on J, contradicting Lemma Q.

Case 2. δ is an arc cd, where c lies within α, d lies within γ. If $A-\delta$ is connected, let pq be an arc in this set joining points of α on opposite sides of c. If η_1 is an arc of γ containing d but not touching pq, let the arc rs join η_1 to β in B; then the arcs pq and $cdrs$ contradict Lemma Q.

Case 3. The end points c and d of δ lie within $\gamma=ab$, say in the order $acdb$. If $A-\delta$ is connected, let pq be an arc in this set joining a point p within α to a point q in γ between c and d. If η_1 is an arc of γ containing q but not touching δ, let r_1s_1 be an arc in B joining η_1 to a point s_1 within β.

The arcs acr_1 of γ and r_1s_1 form an arc acr_1s_1 crossing J; hence

$$R - acr_1s_1 = C_1 + C_2,$$

where C_1 contains the open arc $<as_1>$ of β, and C_2 contains b and points connected with b. As r_1s_1 lies in B, $A'\subset C_1+C_2$; the connected set $A'+b$ lies thus in C_2. If η_2 is an arc of γ containing c but not touching η_1, and r_2s_2 is an arc in C_1 joining η_2 to a point s_2 of β between a and s_1, then $\eta_2+r_2s_2$ does not touch pqr_1s_1, and has only the point c in common with δ.

Similarly, if η_3 is an arc of γ containing d but not touching η_1, there is an arc r_3s_3 in $R-bdr_1s_1$ such that r_3 lies in η_3, s_3 lies in β between s_1 and b, and $\eta_3+r_3s_3$ does not touch pqr_1s_1 and has only the point d in common with δ. The arc r_3s_3 does not touch r_2s_2, as it lies in C_2. Thus the two arcs pqr_1s_1 and $s_2r_2cdr_3s_3$ $(cd=\delta)$ contradict Lemma Q.

Case 4. The same as Case 3, except that $c=a$ or $d=b$, say the latter. Then, in the notation of Case 3, the arcs pqr_1s_1 and s_2r_2cb $(cb=\delta)$ contradict Lemma Q.

This completes the proof that A and J_A (B and J_B) satisfy the conditions of Theorem I.

10. **The cutting up of** R. We are concerned with the following lemma.

LEMMA R. *R may be cut into a finite number of pieces of arbitrarily small diameter.*

Given an $\epsilon>0$, choose $\delta<\epsilon$ so as to satisfy the requirement in Lemma F. Suppose R is cut up so that the diameter of the boundary of each piece is $<\delta$. Then each piece is of diameter $<3\epsilon$. For otherwise there is a point q of some piece R_i at a distance $\geqq\epsilon$ from its boundary J_i. Let p be a point of J_i, and q', a point of $R-R_i$ at a distance $\geqq\epsilon$ from p. Every arc from q to q' must cut the boundary J_i of R_i and thus must pass within δ of p, contradicting Lemma F.

The lemma thus follows from

LEMMA S. *Given a $\delta>0$, R can be cut up so that the diameter of the boundary of each piece is $<\delta$.*

Express R as the union of a finite number of continua:

$$R = K_1 + K_2 + \cdots + K_m,\ \delta(K_i) < \delta/2.$$

We shall cut up R in such a manner that no two of these continua K_i and K_j have points on the boundary of the same piece of R, if $K_i\cdot K_j=0$; the lemma will then follow.

Suppose we have cut R up a certain amount (perhaps not yet at all), into the pieces $R_1, R_2, \cdots, R_n$, with boundaries $J_1, J_2, \cdots, J_n$ (we may have R and J alone). Of course *each boundary J_i separates R_i from the rest of R*. Take any two continua, say K_1 and K_2, with $K_1\cdot K_2=0$, each of which has points on one of these J_i, say J_1. We shall cut R up further so that in the new pieces there is no one (i.e. no *piece*, not merely no *boundary* of a piece) which has any points in common with both K_1 and K_2; then on any further cutting up of R, this will still be true.

Divide the points of J_1 into three sets, as follows. We put a point x into the first set if it lies in K_1, or if following J_1 in both directions we reach points of K_1 before reaching points of K_2; we put x into the second set if the same conditions hold with K_1 and K_2 interchanged; all other points we put into the third set. This set L_3' consists of open intervals of J_1, each being bounded by a point of K_1 on one end and a point of K_2 on the other. The points of the first set together with the points $K_1 \cdot R_1$ form a closed set L_1, and those of the second set together with $K_2 \cdot R_1$ form a closed set L_2. Then $\rho(L_1, L_2) > 0$ as $L_1 \cdot L_2 = 0$, from which follows that there are but a finite number of intervals in L_3'. As K_1 is connected, each component of L_1 has points on J_1, and thus on one of the intervals L_3 of J_1 complementary to the intervals of L_3'. Thus there are a finite number of components $L_{11}, L_{12}, \cdots, L_{1m_1}$ in L_1. Similarly there are a finite number of components $L_{21}, L_{22}, \cdots, L_{2m_2}$ in L_2.

We shall now cut R_1 into a number of pieces, in each of which either K_1 has no points or K_2 has no points. Suppose $L_{31}', \cdots, L_{3m_3}'$ and $L_{31}, \cdots, L_{3m_3}$ are the intervals of L_3' and L_3 respectively, and say they lie in the order L_{31}, L_{31}', L_{32}, L_{32}', $\cdots$, L_{3m_3}, L_{3m_3}' on J_1. If we go around J_1, the intervals of L_3 lie alternately in L_1 and L_2. Starting at L_{31}, which lies in L_{11} say, go around J_1 till we reach another interval L_{3k} in L_{11} (we may have gotten back to L_{31}). Put L_{32}, $L_{3,k-1}$ and all of J_1 between these into a set M_2' (which may be L_{32} alone), and put L_{3k}, L_{31}, and all of J_1 between these on the other side from L_{32} into a set M_1' (which may be L_{31} alone). L_{31}' and $L_{3,k-1}'$ are the two intervals of J_1 complementary to M_1' and M_2'.

No set L_{1i} or L_{2j} has points in both M_1' and M_2'. This follows for L_{11} by construction. If it were false for some other set, say L_{1s}, then L_{1s} would have points on two intervals L_{3p} and L_{3q} separated by L_{31} and L_{3k} on J_1. Now $L_{11} \cdot L_{1s} = 0$, hence $\rho(L_{11}, L_{1s}) > 0$. As R_1 is a continuous curve, there are continuous curves L_{11}^* and L_{1s}^* in R_1 containing L_{11} and L_{1s} and such that $L_{11}^* \cdot L_{1s}^* = 0$ (see Lemma C). These sets are arcwise connected, and we can draw arcs contradicting Lemma Q.

Let M_1 be M_1' plus all components L_{1i} and L_{2j} containing points of M_1', and define M_2 similarly. Then M_1 and M_2 are closed, $M_1 \cdot M_2 = 0$, and $M_1 + M_2 \supset L_1 + L_2$. By Lemma L we can draw an arc γ_1 from L_{31}' to $L_{3,k-1}'$ which has no points in M_1 or in M_2. R_1 is thus cut into two pieces, in each of which there is at least one component L_{1i} or L_{2j}; for one contains L_{11}, and the other contains that L_{2j} containing L_{32}. Thus in each piece there are less than $m_1 + m_2$ components, the number in R_1.

If one of the resulting pieces contains more than one component, we cut it up, etc. Finally each new piece of R_1 has points of only one component, and thus K_1 and K_2 are separated in R_1. We now separate K_1 and K_2 in each other piece R_i of R also. This is possible, for if K_i $(i = 1, 2)$ has points in any R_k, it also has points on J_k.

If now there are any other two of the continua K_i and K_j, $K_i \cdot K_j = 0$, each of which has points on some new J_k, we cut R further till this is no longer true, etc. This completes the proof.

11. **The homeomorphism.** Cut R into pieces of diameter $<$ some σ. We make corresponding cuts in R' as follows. The first arc γ drawn in R cuts R into the two pieces R_1 and R_2 with boundaries J_1 and J_2 say. Draw any arc γ' crossing J' in R', cutting R' into the pieces R_1' and R_2' with boundaries J_1' and J_2'. We note that $J_1'+J_2'$ is homeomorphic with J_1+J_2, with J_k' corresponding to J_k, $k=1, 2$. Say γ_1 is an arc in R_1, cutting R_1 into pieces R_{11} and R_{12} with boundaries J_{11} and J_{12}. If a_1 and b_1 are the end points of γ_1, let a_1^* and a_2^* be the corresponding points of J_1' in the above homeomorphism. Draw an arc γ_1' crossing J_1' in R_1', with end points a_1' and b_1' close to a_1^* and b_1^* respectively (Lemma P); R_1' is divided thereby into the pieces R_{11}' and R_{12}' with boundaries J_{11}' and J_{12}'. Moreover, $J_{11}'+J_{12}'+J_2'$ is homeomorphic with $J_{11}+J_{12}+J_2$, with boundaries with the same subscripts corresponding.

In general, suppose $R_{i_1 i_2 \cdots i_m}$ is a piece that is present after R is cut a certain amount, and say the arc $\gamma_{i_1\cdots i_m}$ divides this set into the pieces $R_{i_1\cdots i_m 1}$ and $R_{i_1\cdots i_m 2}$, with boundaries $J_{i_1\cdots i_m 1}$ and $J_{i_1\cdots i_m 2}$. If $a_{i_1\cdots i_m}$ and $b_{i_1\cdots i_m}$ are the end points of $\gamma_{i_1\cdots i_m}$, let $a_{i_1}^*{}_{\cdots i_m}$ and $b_{i_1}^*{}_{\cdots i_m}$ be the corresponding points on $J_{i_1\cdots i_m}'$ in the homeomorphism we have already. Draw an arc $\gamma_{i_1}'{}_{\cdots i_m}$ crossing $J_{i_1}'{}_{\cdots i_m}$, with end points $a_{i_1}'{}_{\cdots i_m}$ and $b_{i_1}'{}_{\cdots i_m}$ close to the above points, dividing $R_{i_1}'{}_{\cdots i_m}$ into the pieces $R_{i_1}'{}_{\cdots i_m 1}$ and $R_{i_1}'{}_{\cdots i_m 2}$, with boundaries $J_{i_1}'{}_{\cdots i_m 1}$ and $J_{i_1}'{}_{\cdots i_m 2}$. The set of boundaries with primes is now homeomorphic with the set of boundaries without primes, boundaries with the same subscripts corresponding. We note that *if $R_{i_1\cdots i_m}$ and $R_{j_1\cdots j_m}$ have common points, then $R_{i_1}'{}_{\cdots i_m}$ and $R_{j_1}'{}_{\cdots j_m}$ have common points, and conversely.*

Having cut R into pieces of diameter $<\sigma$ and having cut R' in a corresponding fashion, we now cut each piece of R' into pieces of diameter $<\sigma/2$ and cut each piece of R in a corresponding fashion. Next we cut each resulting piece of R into pieces of diameter $<\sigma/4$, etc. Now for any $\epsilon>0$ there is an m such that

$$\delta(R_{i_1\cdots i_m}) < \epsilon, \quad \delta(R'_{i_1\cdots i_m}) < \epsilon,$$

for any m-fold subscript.

We now establish the homeomorphism between R and R'. Let p be any point of R. It lies in either R_1 or R_2 (perhaps in both), say in R_{i_1}. Then it lies in either $R_{i_1 1}$ or $R_{i_1 2}$ (perhaps in both), say in $R_{i_1 i_2}$, etc. Thus we have a sequence of pieces

$$R \supset R_{i_1} \supset R_{i_1 i_2} \supset \cdots \supset p.$$

The corresponding pieces in R' have a single limit point:

$$R' \supset R'_{i_1} \supset R'_{i_1 i_2} \supset \cdots \supset p'.$$

This point p' we let correspond to p.

If there are different sequences of pieces in R containing p, we have different sequences in R' defining points p'. However, all these points p' are the same. For if R, R_{i_1}, $R_{i_1 i_2}$, $\cdots$, and R, R_{j_1}, $R_{j_1 j_2}$, $\cdots$, are two sequences containing p, then each piece $R_{i_1 \cdots i_m}$ has points in common with $R_{j_1 \cdots j_m}$, namely, the point p; hence, as we saw above, $R'_{i_1 \cdots i_m}$ and $R'_{j_1 \cdots j_m}$ have common points. Thus the corresponding sequences in R' close down on a single point. Similarly, to each point p' in R' corresponds a single point p in R.

Finally, the correspondence is continuous. For take a point p in R and an $\epsilon > 0$. Let p' be the corresponding point in R', and choose an m so that $\delta(R'_{i_1 \cdots i_m}) < \epsilon$ for all m-fold subscripts. Consider all the $R_{i_1 \cdots i_m}$ with m-fold subscripts which contain p; these include all points of R in some $V_\delta(p)$. Then if $q \subset V_\delta(p)$, the corresponding point q' is in $V_\epsilon(p')$, and the continuity is established. This completes the proof of Theorem I.

12. **Proof of Theorem II.** Let I be a circle in the plane, and let S be I plus its interior. S is self-compact, connected and locally connected, and is thus a continuous curve. That $I \sim 0$ in S follows from Lemma K.†

To show that I is irreducibly ~ 0 in S, suppose that $I \sim 0$ in S', a proper closed subset of S; we can suppose that S' is a continuous curve. Let p be a point of S not in S', and let $V_\delta(p)$ have no points in S'. Let ab be a segment of a straight line passing through p with its ends on I. Let a_1b_1 and a_2b_2 be parallel segments enclosing ab, and lying at a distance δ from ab. Then in that portion of S' between a_1b_1 and a_2b_2, the (short) arcs a_1a_2 and b_1b_2 are not connected. But if C and D are those parts of S' outside a_1b_1 and a_2b_2, by Lemma L we can draw an arc joining a_1a_2 to b_1b_2 in $S' - (C + D)$, a contradiction.

Finally, that an arc crossing I in S divides S is a special (and easily proved) case of the Jordan theorem. This completes the proof.

13. **The Jordan theorem.** Let J be a simple closed curve in the plane. Let I be a circle containing J in its interior. Draw two non-intersecting line segments from I to J. $S = I$ plus its interior is thus cut into three closed 2-cells, one of which, say R, has the boundary J. Then $R - J$ is the inside of J. The points of J are obviously accessible from either side.

† For S is a closed 2-cell.

PRINCETON UNIVERSITY,
PRINCETON, N. J.